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ON LATERAL BUCKLING OF END-LOADED CANTILEVER BEAMS.(U)
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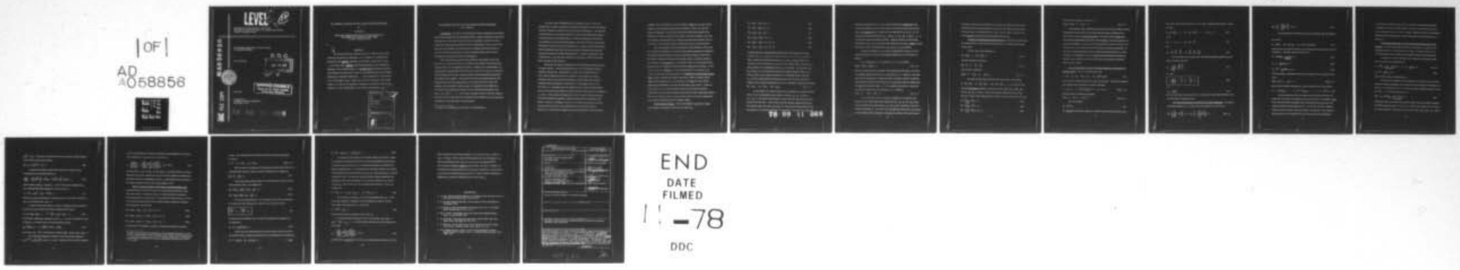

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by E. Reissner

July 1978

Prepared for
OFFICE OF NAVAL RESEARCH
Washington, D.C.

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ABSTRACT

The classical problem of lateral buckling of cantilever beams with transverse end-load is re-examined, as an example of a problem fully governed by the intrinsic equations of Kirchhoff's curved beam theory. — It is shown that a suitable non-dimensionalization of the differential equations of the problem leads to a straightforward perturbation solution, with leading and second-order terms of the expansion having well-defined differences in physical significance. — The equations of a recent extension of Kirchhoff's theory, which take account of transverse shear deformation, are used for the purpose of obtaining a numerical result for the influence of shear deformability on the lateral buckling load.

ON LATERAL BUCKLING OF END-LOADED CANTILEVER BEAMS†

By E. Reissner

Introduction. The history of the problem of lateral buckling of transversely loaded beams begins with two fundamental papers, written independently and nearly simultaneously, by A.G.M. Michell [1] and L. Prandtl [2]. Michell as well as Prandtl used appropriate geometrical ad hoc considerations to arrive at a correct physical understanding of the problem and at a buckling load formula which is correct, except for the analysis of certain secondary effects which are of no significance in almost all practical circumstances.

Five years after the publication of Michell's and Prandtl's work it was observed by H. Reissner [3] that the equations of the problem of lateral buckling could be deduced in a straightforward manner, without ad hoc considerations, by an appropriate specialization of Kirchhoff's general theory of space-curved beams, with the analysis of the (two) secondary effects, being automatically included in the analysis of the problem. Beyond making the above important advance in the analysis of the lateral buckling problem, H. Reissner went on to reduce the problem of the end-loaded cantilever beam to a boundary value problem for a third order linear differential equation, and to a buckling load equation of the form $c_1P + c_2P^2 + \ldots = 1$. In arriving at the above result H. Reissner neglects one of the two secondary effects and indicates that he intends to give the numerical consequences of his formula in a different place.

† A report on work supported by the Office of Naval Research.

Ten years after the publication of H. Reissner's note, in 1914, his assistant M.K. Grober reconsiders the problem [4] "based on some calculations which Mr. Reissner turned over to me for further development." The principal result of Grober's work is the derivation of a third-order differential equation with full inclusion of secondary effects, and with a buckling formula given by the vanishing of a second order determinant, with each of the four terms in the determinant a power series in the buckling load. Grober concludes his analysis with the statement that he will "as soon as possible evaluate some numerical cases and compare the results with experiment." Grober was killed in action in World War I, very soon after completion of this paper, and thus prevented from carrying out his intentions.

Thereafter the number of publications on the problem of lateral buckling increases steadily. From among this literature two contributions should be mentioned specifically. One of these is a paper by K. Federhofer [5], in 1931, which includes the numerical evaluation of H. Reissner's buckling equation for the case of a narrow rectangular-cross section beam (with the secondary effect amounting to 7.8 percent for a width-depth ratio of 1/5). The other is a recent paper by D.H. Hodges and D.A. Peters [7] which undertakes to re-examine the problem once again ab initio on the basis of H. Reissner's approach. In the process of doing this the authors rederive Grober's general third-order differential equation, unaware of this earlier contribution to the subject. However, in addition, the authors reconsider the problem of systematically determining first-order approximations for secondary effects. They find, in what must be considered an important advance in the field of this problem, that a systematic first-order analysis of <u>both</u> second-order effects comes out to be actually simpler than the analysis in which one of the two effects is neglected, with the basic third-order differential equation of the problem now having an explicit first integral, leaving the problem in the form of a boundary value problem for a second-order differential equation, just as for the case of the problem without consideration of any secondary effects.

The main purposes of the present paper are the following. (1) We wish to re-consider the problem on the basis of Kirchhoff's equations of equilibrium for finitely deforming rods in such a way that full advantage is taken of the fact that the boundary conditions of the problem allow a complete solution of the equations of an <u>intrinsic</u> form of the theory, that is of a formulation of the theory without any regard to the form of strain displacement relations. (2) We wish to show that a <u>suitable</u> non-dimensionalization of the equations of the theory indicates the evident possibility of a <u>straightforward perturbation expansion</u>, in such a way that the results of the theory without secondary effects appear as the leading terms in the expansion, with <u>both</u> secondary effects appearing systematically in second and still higher-order terms. (3) We use the equations of a recently developed extension of Kirchhoff's equations, which takes account of axial extension and transverse shear deformation effects [6], for the purpose of determining the effect of transverse shear deformation on the lateral buckling load of the cantilever beam.

<u>Formulation of Problem.</u> We write Kirchhoff's equations for finite deformations of originally straight beams in the form

$$P_t' - \varkappa_1 P_1 - \varkappa_2 P_2 + p_t = 0 \quad , \tag{1a}$$

$$P_1' + \varkappa_1 P_t - \varkappa_t P_2 + p_1 = 0 \quad , \tag{1b}$$

$$P_2' + \varkappa_2 P_t + \varkappa_t P_1 + p_2 = 0 \quad , \tag{1c}$$

$$M_t' + \varkappa_1 M_2 - \varkappa_2 M_1 + m_t = 0 \quad , \tag{2a}$$

$$M_1' + \varkappa_2 M_t - \varkappa_t M_2 + P_1 + m_1 = 0 \quad , \tag{2b}$$

$$M_2' - \varkappa_1 M_t + \varkappa_t M_1 + P_2 + m_2 = 0 \quad . \tag{2c}$$

In these equations primes indicate differentiation with respect to an axial coordinate x, P_t, P_1 and P_2 are forces acting over the cross section of the beam, tangent and normal to the center line, with P_1 and P_2 being in the directions of the principal axes y_1 and y_2 of the cross section, and with p_t, p_1 and p_2 being the corresponding surface force intensities. Furthermore, M_t, M_1 and M_2 are cross sectional twisting and bending moments, with m_t, m_1 and m_2 being surface moment intensities corresponding thereto, and with $\varkappa_t$, $\varkappa_1$ and $\varkappa_2$ being twisting and bending strains which are here taken to be related to the twisting and bending moments by constitutive equations of the form

$$M_t = D_t \varkappa_t \ , \quad M_1 = D_1 \varkappa_1 \ , \quad M_2 = D_2 \varkappa_2 \quad . \tag{3a, b, c}$$

In what follows we restrict attention to the problem of a cantilever beam which is free of distributed surface loads, so that $p_t = p_1 = p_2 = 0$ and $m_t = m_1 = m_2 = 0$, and which is acted upon by a force and by a moment at the unsupported end of the beam. Of the various possible loading conditions which may be subsumed under the above description we will be concerned specifically with the problem of a transverse end force P oriented in a direction which coincides with the principal direction y_1 of the cross section of the undeformed beam. It is evident that the problem as described is of such nature that one possible state, the unbuckled state, involves no more than the two forces P_1, P_t, the moment M_1 and the bending strain $\varkappa_1$, with $P_2 = 0$, $M_2 = M_t = 0$ and $\varkappa_t = \varkappa_2 = 0$, and with the associated buckling problem being the problem of determining the smallest value of P which allows the existence of alternate, buckled, states with some or all of the quantities P_2, M_2, M_t, $\varkappa_2$, $\varkappa_t$ nonvanishing.

Boundary conditions for the system of differential equations (1) to (3) which correspond to the prescribed loading condition may be formulated as follows.

We evidently have at the loaded end, $x = 0$, the conditions

$$M_1(0) = M_2(0) = M_t(0) = 0 \quad . \tag{4a, b, c}$$

Since we do not know the orientation of the loaded end of the beam, we cannot say anything about the values of P_1, P_2 and P_t for $x = 0$. However, the condition that the direction of P remains the same, no matter what the orientation of the loaded end cross section might be, in conjunction with the condition that the supported end of the beam, $x = L$, is assumed to be built-in, means that we know the values of P_1, P_2 and P_t for $x = L$, as follows

$$P_1(L) = P \quad , \quad P_2(L) = P_t(L) = 0 \quad . \tag{5a, b, c}$$

We note that the number of boundary conditions corresponds to the number of first order differential equations for P_1, P_2, P_t, M_1, M_2 and M_t which is obtained upon eliminating $\varkappa_1$, $\varkappa_2$ and $\varkappa_t$ from equations (1) and (2) by means of equations (3). We further note specifically that the above formulation of the problems holds, without any reference to relations which exist between the strain components $\varkappa$ and whatever description we might choose for translational and rotational displacement components of the elements of the beam. In other words, our problem has been stated entirely within the frame work of the intrinsic equations of one-dimensional beam theory.

Derivation of Buckling Differential Equations. We begin by considering the unbuckled state, with overbars designating forces, moments and strains of this state.

We have then, from equations (1)

$$\overline{P}_t' - \overline{\varkappa}_1 \overline{P}_1 = 0 \ , \ \overline{P}_1' + \overline{\varkappa}_1 \overline{P}_t = 0 \ , \tag{6a, b}$$

and from equations (2) and (3),

$$\overline{M}_1' + \overline{P}_1 = 0 \ , \ \overline{M}_1 = D_1 \overline{\varkappa}_1 \ , \tag{7a, b}$$

with boundary conditions

$$\overline{M}_1(0) = 0 \ , \ \overline{P}_1(L) = P \ , \ \overline{P}_t(L) = 0 \ . \tag{8a, b, c}$$

We obtain equations governing the onset of buckling, upon setting

$$P_t = \overline{P}_t + \Delta P_t \ , \ P_1 = \overline{P}_1 + \Delta P_1 \ , \ M_1 = \overline{M}_1 + \Delta M_1 \ , \ \varkappa_1 = \overline{\varkappa}_1 + \Delta \varkappa_1 \tag{9 a-d}$$

and upon linearizing equations (1) and (3) in terms of ΔP_t, ΔP_1, ΔM_1, $\Delta \varkappa_1$ and P_2, M_2, M_t, $\varkappa_2$ and $\varkappa_t$. Of the six equations obtained in this way only three are needed, those following from equations (1a), (2a) and (2c) in the form

$$P_2' + \underline{\underline{\overline{P}_t \varkappa_2}} + \overline{P}_1 \varkappa_t = 0 \ , \tag{10a}$$

$$M_t' + \underline{\overline{\varkappa}_1 M_2} - \overline{M}_1 \varkappa_2 = 0 \ , \tag{10b}$$

$$M_2' - \underline{\overline{\varkappa}_1 M_t} + \overline{M}_1 \varkappa_t + P_2 = 0 \ . \tag{10c}$$

The associated boundary conditions are

$$M_2(0) = M_t(0) = 0 \ , \quad P_2(L) = 0 \ . \tag{11a, b, c}$$

Note that the singly underlined terms in (10) describe the effect of initial deformations on the process of buckling while the doubly underlined term describes the effect of finite deformation in the analysis of the initial state. For practical applications both effects in the analysis of the given problem are generally negligible. In the work of H. Reissner [3] the effect of the $\bar{\kappa}_1$-terms is taken into account and the effect of the $\overline{P}_t$-term is explicitly neglected. In the work of Hodges and Peters [7], it is observed that both effects are of the same order of magnitude in terms of appropriate dimensionless parameters but that, numerically, the effect of the $\overline{P}_t$-term is only about one-fifth the effect of the $\bar{\kappa}_1$-terms.

Non-Dimensionalization and Perturbation Expansion for Equations of Unbuckled State. We set in equations (6) to (8)

$$x = L\xi \ , \quad \overline{P}_1 = Pp \ , \quad \overline{M}_1 = PLm \ , \quad \overline{P}_t = (PL^2/D_1)Pq \ , \tag{12}$$

and we indicate differentiation with respect to ξ by dots. With this the differential equations (6) and (7) may be written in the form

$$\dot{q} - mp = 0 \ , \quad \dot{p} + (PL^2/D_1)^2 mq = 0 \ , \quad \dot{m} + p = 0 \ , \tag{13a, b, c}$$

and the boundary conditions (8) become

$$m(0) = 0 \ , \quad p(1) = 1 \ , \quad q(1) = 0 \ . \tag{14a, b, c}$$

We now consider

$$\eta_1 = PL^2/D_1 \tag{15}$$

as a small parameter and expand the solution of (13) and (14) in powers of η_1.

The result of the simple calculation, to the degree needed in what follows, comes out to be

$$p = p_0 + \eta_1^2 p_1 + \dots \quad , \quad m = m_0 + \eta_1^2 m_1 + \dots \quad , \quad q = q_0 + \dots \quad , \tag{16}$$

where

$$p_0 = 1 \quad , \quad m_0 = -\xi \quad , \quad q_0 = \tfrac{1}{2}(1 - \xi^2) \quad , \tag{17}$$

and

$$p_1 = -\frac{1}{8} + \frac{\xi^2}{4} - \frac{\xi^4}{8} \quad , \quad m_1 = \frac{\xi}{8} - \frac{\xi^3}{12} + \frac{\xi^5}{40} \quad . \tag{18}$$

For what follows it is important to note that the small parameter η_1 may also be written in the form

$$\eta_1 = \frac{PL^2}{\sqrt{D_2 D_t}} \sqrt{\frac{D_t}{D_1} \frac{D_2}{D_1}} \quad , \tag{19}$$

or, with

$$\boxed{\sigma = \frac{PL^2}{\sqrt{D_2 D_t}} \quad , \quad \frac{D_t}{D_1} = \eta_t \quad , \quad \frac{D_2}{D_1} = \eta_2 \quad ,} \tag{20}$$

as

$$\eta_1 = \sigma\sqrt{\eta_t \eta_2} \quad , \tag{21}$$

where, as is known from previous work, the value of σ for which buckling occurs is of the order of magnitude unity.

Non-Dimensionalization of Equations for On-Set of Buckling. We begin by rewriting equations (10), in conjunction with equations (3), in the form

$$P_2' + \underline{\underline{\overline{P}_t \frac{M_2}{D_2}}} + \overline{P}_1 \frac{M_t}{D_t} = 0 \quad , \quad M_t' - \left(1 - \frac{D_2}{D_1}\right)\overline{M}_1 \frac{M_2}{D_2} = 0 \quad , \tag{22a, b}$$

$$M_2' + \left(1 - \frac{D_t}{\underline{D_1}}\right)\overline{M}_1 \frac{M_t}{D_t} + P_2 = 0 \quad . \tag{22c}$$

We next introduce into these the contents of equation (12), and furthermore write

$$P_2 = Pg(\xi) \quad , \quad M_2 = PLf_2(\xi) \quad , \quad M_t = (PL^2/D_2)PLf_t(\xi) \quad , \tag{23}$$

with the choice of the factor PL^2/D_2 in the expression for M_t being of particular importance. With this equations (22) become

$$g' + \sigma^2[p(\xi)f_t + \underline{\underline{\eta_t q(\xi)f_2}}] = 0 \quad , \tag{24a}$$

$$f_t' - (1 - \underline{\eta_2})m(\xi)f_2 = 0 \quad , \tag{24b}$$

$$f_2' + \sigma^2(1 - \underline{\eta_t})m(\xi)f_t + g = 0 \quad , \tag{24c}$$

with the boundary conditions for this system following from equations (4) and (5) as

$$f_2(0) = f_t(0) = 0 \quad , \quad g(1) = 0 \quad , \tag{25a, b, c}$$

and with the coefficient functions p, q and m following from (16) to (20) as

$$p = 1 + \sigma^2\eta_2\eta_t p_1 + \ldots \quad , \quad m = -\xi + \sigma^2\eta_2\eta_t m_1 + \ldots \quad , \quad q = \tfrac{1}{2} - \tfrac{1}{2}\xi^2 + \ldots \tag{26}$$

We specifically note from the appearance of equations (24) that, as apparently first observed by Hodges and Peters [7], there are altogether <u>three</u> terms which determine the effect of the small parameters η_2 and η_t on the smallest possible nonvanishing value of σ. We further note that it appears likely that the third order eigenvalue problem (24) and (25), with coefficient functions given in accordance with (13) and (14) without any assumptions concerning the smallness of η_2 and η_t, would offer no particular difficulties in regard to a direct numerical solution. However, we will limit ourselves here to seeing in which way a direct solution, without use of such computational facilities, becomes possible upon explicit utilization of the assumptions $\eta_t \ll 1$ and $\eta_2 \ll 1$.

Perturbation Expansion for the Solution of the Characteristic Value Problem. It is clear from (24) to (26) that the functions g, f_t and f_2 may be expanded in powers of the parameters η_t and η_2. We limit ourselves here to a determination of the zeroth and first degree terms in these expansions.

As long as we restrict attention to first degree terms equations (24) may be written in the simplified form

$$g^{\cdot} + \sigma^2[f_t + \underline{\underline{\tfrac{1}{2}\eta_t(1 - \xi^2)f_2}}] = 0 \ , \quad f_t^{\cdot} + (1 - \underline{\eta_2})\xi f_2 = 0 \ , \qquad \text{(27a, b)}$$

$$f_2^{\cdot} - \sigma^2(1 - \underline{\eta_t})\xi f_t + g = 0 \ , \qquad \text{(27c)}$$

again with the boundary conditions (25).

The third order problem (27) and (25) may be reduced to a second order problem, through recognition of the existence of a first integral of the system, as follows. We multiply (27a) by a factor ξ and add the resulting relation to equation (27c). In this way there follows first

$$(\xi g)^{\cdot} + f_2^{\cdot} + \eta_t\sigma^2[\underline{\xi f_t} + \underline{\underline{\tfrac{1}{2}(\xi - \xi^3)f_2}}] = 0 \ . \qquad \text{(28)}$$

Having the factor η_t in front of the bracket in (28), we may now utilize equations (27) without η_2 and η_t-terms, in order to transform the contents of the bracket in (28) advantageously. Using (27b) we obtain $\xi f_t + \frac{1}{2}(\xi - \xi^3)f_2 = \xi f_t - \frac{1}{2}(1 - \xi^2)f_t^{\cdot} =$

$\frac{1}{2}[(\xi^2 - 1)f_t]^{\cdot}$. Therewith, and with the first two conditions in (25) we deduce from (28) the first-integral relation

$$\xi g + f_2 + \tfrac{1}{2}\eta_t \sigma^2(\xi^2 - 1)f_t = 0 \quad . \tag{29}$$

Having (29) we obtain a differential equation for g alone by first combining (27a, b) and (29) in the form

$$-\left(\frac{g^{\cdot}}{\sigma^2}\right)^{\cdot} - \eta_t\left(\frac{1-\xi^2}{2}f_2\right)^{\cdot} + \xi\left[-\xi g - \sigma^2\eta_t\frac{\xi^2-1}{2}f_t\right] + \eta_2\xi^2 g = 0 \quad , \tag{30}$$

and by setting in this $f_2 = -\xi g$ and $f_t = -g^{\cdot}/\sigma^2$ in the terms multiplied by η_t. The resulting differential equation for g comes out to be

$$g^{\cdot\cdot} + \sigma^2[(1-\eta_2)\xi^2 - \tfrac{1}{2}\eta_t(1 - 3\xi^2)]g = 0 \quad , \tag{31}$$

with the two associated boundary conditions being, in accordance with (25) and (27), the conditions $g^{\cdot}(0) = g(1) = 0$.

In order to solve the problem as stated, including first-order effects in η_2 and η_t, we use ordinary perturbation expansions of the form

$$g = g_0 + \eta_2 g_2 + \eta_t g_t + \ldots \quad , \quad \sigma^2 = \sigma_0^2(1 + c_2\eta_2 + c_t\eta_t + \ldots) \quad . \tag{32}$$

It is evident, without any calculations, that $c_2 = 1$. In order to obtain the values of σ_0^2 and c_t, we deduce from (31) the differential equations

$$g_0^{\cdot\cdot} + \sigma_0^2\xi^2 g_0 = 0 \quad , \quad g_t^{\cdot\cdot} + \sigma_0^2\xi^2 g_t = [h(\xi) - c_t\xi^2]g_0 \quad , \tag{33}$$

where $h(\xi) = \frac{1}{2}(1 - 3\xi^2)$, with boundary conditions $g_0^{\cdot}(0) = g_t^{\cdot}(0) = g_0(1) = g_t(1) = 0$.

The well-known appropriate solution of the zeroth-order equation is $g_0 = \xi^{1/2}J_{-1/4}\left(\frac{1}{2}\sigma_0\xi^2\right)$, where $\sigma_0 \approx 4.0126$. Solution of the first-order equation in (33) by the method of variation of parameters and satisfaction of the boundary conditions for g_t then gives as the value of c_t,

$$c_t = \frac{\int_0^1 h g_0^2 d\xi}{\int_0^1 \xi^2 g_0^2 d\xi} = \frac{1}{2} \frac{\int_0^1 \left[J_{-1/4}\left(\frac{1}{2}\sigma_0 \xi^2\right)\right]^2 d\xi}{\int_0^1 \left[\xi J_{-1/4}\left(\frac{1}{2}\sigma_0 \xi^2\right)\right]^2 d\xi} - \frac{3}{2} \approx 1.285 \ , \tag{34}$$

and therewith $\sigma \approx \sigma_0(1 + 0.5\eta_2 + 0.64\eta_t)$ where, it should be noted, the correct numerical value of the coefficients of η_t has first been obtained by Hodges and Peters [7], with the corresponding value of c_t, which follows upon omission of the doubly underlined terms in (27) and (28) being $1.645^\dagger$.

<u>Effect of Transverse Shear Deformation on Lateral Buckling Load.</u>

A determination of the effect of transverse shear may be based on an extension of Kirchhoff's equations for beams, in which the effect of transverse shear deformations γ_1 and γ_2 and of a longitudinal extensional strain γ_t is taken in account of by replacing the three Kirchhoff moment equilibrium equations (2) by equations of the form [6],

$$M_t' + \varkappa_1 M_2 - \varkappa_2 M_1 + \gamma_1 P_2 - \gamma_2 P_1 + m_t = 0 \ , \tag{35a}$$

$$M_1' + \varkappa_2 M_t - \varkappa_t M_2 + (1+\gamma_t) P_1 - \gamma_1 P_t + m_1 = 0 \ , \tag{35b}$$

$$M_2' - \varkappa_1 M_t + \varkappa_t M_1 + (1+\gamma_t) P_2 - \gamma_2 P_t + m_2 = 0 \ , \tag{35c}$$

in conjunction with additional constitutive equations involving the quantities

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$\dagger$ In order to see the numerical significance of the improvement in [7], we note that for a homogeneous narrow rectangular cross section, with Poissons's ratio $\nu = 1/3$ we have, when $\eta_t = 0.1$, that $\sigma \approx 1.097\sigma_0$, whereas the corresponding result without consideration of the doubly underlined terms comes out to be $\sigma \approx 1.115\sigma_0$.

γ and P. For what follows we take these additional constitutive equations in the form

$$\gamma_t = 0 \ , \ \gamma_1 = C_1 P_1 \ , \ \gamma_2 = C_2 P_2 \ . \tag{36a, b, c}$$

With the above the equations corresponding to equations (6) to (8) for the unbuckled state remain as before except that equation (7a) is replaced by

$$\overline{M}_1' + \overline{P}_1 - \overline{\gamma}_1 \overline{P}_t = 0 \ . \tag{7a*}$$

At the same time equation (10a) for the buckled state remains as before while equations (10b, c) are replaced by

$$M_t' + \overline{\varkappa}_1 M_2 - \varkappa_2 \overline{M}_1 + \overline{\gamma}_1 P_2 - \gamma_2 \overline{P}_1 = 0 \ , \tag{10b*}$$

$$M_2' - \overline{\varkappa}_1 M_t + \varkappa_t \overline{M}_1 + P_2 - \gamma_2 \overline{P}_t = 0 \ . \tag{10c*}$$

We now non-dimensionalize, as in equation (12) and (23), and introduce two transverse shear deformation parameters λ_1 and λ_2 of the form

$$\boxed{\frac{C_1 D_1}{L^2} = \lambda_1 \ , \ \frac{C_2 D_2}{L^2} = \lambda_2 \ .} \tag{37}$$

We then have that equations (13a, b) remain unchanged while equation (13c) is replaced by

$$m^{\cdot} + (1 - \eta_2 \eta_t \sigma^2 \lambda_1 q) p = 0 \ . \tag{13c*}$$

Of the three non-dimensionalized buckling equations (24) we have that one of them, (24a), remains unchanged while the remaining two are replaced by

$$f_t^{\cdot} - (1 - \eta_2) m f_2 - (\lambda_2 - \eta_2 \lambda_1) p g = 0 \ , \tag{24b*}$$

$$f_2^{\cdot} + \sigma^2(1 - \eta_t)mf_t + (1 - \eta_2\sigma^2\lambda_2 q)g = 0 \quad . \tag{24c*}$$

In considering the problem of solving the system (13) and (24), subject to the boundary conditions (14) and (25) we note the possibility that the transverse shear parameters λ_i may, for sandwich-type beams, be quantities of order of magnitude unity. In what follows we will limit ourselves to a solution of the problem for the case that both λ_1 and λ_2, as well as η_t and η_2, are small compared to unity. We may then neglect products of these parameters and evaluate the effect of nonvanishing λ_i as one which is additive to the effects of η_t and η_2. With this we have that equations (24a) and (24b*, c*) may be simplified to

$$g^{\cdot} + \sigma^2 f_t = 0 \quad , \quad f_t^{\cdot} + \xi f_2 - \lambda_2 g = 0 \quad , \quad f_2^{\cdot} - \sigma^2\xi f_t + g = 0 \quad . \tag{38}$$

We now obtain, as before, a first integral relation, $\xi g + f_2 = 0$, and we use this relation to transform the second equation in (38) to a second-order differential equation for g, of the form

$$g^{\cdot\cdot} + \sigma^2(\xi^2 + \lambda_2)g = 0 \quad , \tag{39}$$

again with the boundary conditions $g^{\cdot}(0) = g(1) = 0$.

We again expand the solution of this, in the form $g = g_0 + \lambda_2 g_\lambda + \ldots$ and $\sigma^2 = \sigma_0^2(1 + c_\lambda\lambda_2 + \ldots)$, and then obtain in the same way as in going from (31) to (34)

$$c_\lambda = - \frac{\int_0^1 \left[J_{-1/4}\left(\frac{1}{2}\sigma_0\xi^2\right)\right]^2 d\xi}{\int_0^1 \left[\xi J_{-1/4}\left(\frac{1}{2}\sigma_0\xi^2\right)\right]^2 d\xi} \approx -5.57 \quad , \tag{40}$$

and therewith a <u>reduction</u> of the value of the buckling load parameter due to the effect of transverse shear deformability, in accordance with the relation $\sigma \approx \sigma_0(1 - 2.785\,\lambda_2)$. Given a beam with homogeneous narrow rectangular cross section of thickness 2c the value of λ_2 comes out to be $(2E/5G)(c^2/L^2)$. For a narrow rectangular <u>sandwich</u> cross section, with shear resistant core of thickness 2c enclosed between two face sheets of thickness t, the parameter λ_2 is given by the expression $(E_f/G_c)(ct/L^2)$, with the evident possibility of a significant λ_2-effect for sufficiently large values of E_f/G_c.

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

6 ON LATERAL BUCKLING OF END-LOADED CANTILEVER BEAMS.

5. TYPE OF REPORT & PERIOD COVERED

9 INTERIM rept.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

10 E./Reissner

8. CONTRACT OR GRANT NUMBER(s)

15 N00014-75-C-0158

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Dept. of Applied Mechanics & Engineering Sciences
University of California, San Diego
La Jolla, California 92093

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

12 18p.

11. CONTROLLING OFFICE NAME AND ADDRESS

Structural Mechanics Branch
Office of Naval Research, Code 474
Arlington, Virginia 22217

12. REPORT DATE

11 Jul[redacted]78

13. NUMBER OF PAGES

15

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cantilever, Lateral buckling, Secondary effects, Perturbation expansions, Transverse shear corrections.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The classical problem of lateral buckling of cantilever beams with transverse end load is re-examined, as an example of a problem fully governed by the intrinsic equations of Kirchhoff's curved beam theory. - It is shown that a suitable non-dimensionalization of the differential equations of the problem loads to a straight-forward perturbation solution, with leading and second-order terms of the expansion having well-defined differences in physical significance. - The equations of a recent extension of Kirchhoff's theory, which take account of transverse shear deformation, are used for the purpose of obtaining a numerical result for the influence of shear deformability on the lateral buckling load.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

407 513

Jan